

# QCD at high baryon density in a random matrix model

 S. Pepin<sup>1,a</sup> and A. Schäfer<sup>2</sup>
<sup>1</sup> Max-Planck-Institut für Kernphysik, Postfach 103980, D-69029 Heidelberg, Germany

<sup>2</sup> Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany

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**Abstract.** A high-density diquark phase seems to be a generic feature of QCD. If so it should also be reproduced by random matrix models. We discuss a specific one in which the random matrix elements of the Dirac operator are supplemented by a finite chemical potential and by non-random elements which model the formation of instanton–anti-instanton molecules. Comparing our results to those found in a previous investigation by Vanderheyden and Jackson we find additional support for our starting assumption, namely that the existence of a high-density diquark phase is common to all QCD-like model.

**PACS.** 11.30.Rd Chiral symmetries – 12.38.Mh Quark-gluon plasma – 12.38.Aw General properties of QCD (dynamics, confinement, etc.)

## 1 Introduction

The understanding of the QCD phase diagram at finite density and temperature is of fundamental interest to investigate phenomena related to heavy-ion collisions and to the physics of neutron stars. While first principles lattice calculations are able to take into account the effects of finite temperature, the addition of a baryonic chemical potential to the lattice action makes the fermionic determinant complex, which prevents reliable Monte Carlo calculations. Resorting to specific QCD models or to an asymptotic QCD expansion for very large temperature or chemical potential  $\mu$  is thus necessary to investigate the finite density domain. These models allow to study the symmetries of the different regions of the phase diagram; of particular interest is the study of the chiral restoration at finite temperature and/or density. Recently, a lot of attention has been focused on the so-called “colour superconductivity”: at large density and low temperature, an arbitrarily weak attractive force makes the Fermi sea of quarks unstable with respect to diquark formation and induces Cooper pairing (diquark condensation) [1–4]. At still higher  $\mu$  this turns into still another, so-called colour-flavour-locked, phase [5]. As already stated, there exists no really reliable QCD technique to analyse these phases, except for completely unrealistically large  $\mu$ . However, all the different approaches used so far yield the same general picture which gives credence to the assumption that it is generic. Some of the different approaches followed are NJL-like models [3], which can be closely related to

models based on the instanton phenomenology [4], the Dyson-Schwinger approach [6], and, at very large  $\mu$ , weak coupling expansions [7]. These calculations agree in their main conclusions and suggest a fairly large gap (on the order of 100 MeV), which could have some observational consequences in neutron stars and possibly even heavy-ion collisions (for recent reviews see [8,9]). One should note however that colour superconductivity in QCD differs from usual superconductivity in the following sense: for large chemical potential, the Renormalisation Group Approach leads to an asymptotic dependence of the gap as a function of the QCD coupling constant of the kind  $\Delta \sim \exp[-c/g]$  [10]. The naive expectation from BCS theory would be in contrast  $\Delta \sim \exp[-c/g^2]$ . This difference in behaviour is due to the long-range nature of the magnetic interaction in QCD: while non-static magnetic modes of the gluon propagator are dynamically screened due to Landau damping, static magnetic modes are not screened. It leads to a considerable enhancement of the QCD superconducting gap.

To test the hypotheses that these new phases are indeed a generic feature of QCD and QCD-like models, such that the insufficiencies of the individual approaches are unimportant for the existence of these phases, Random Matrix Theory (RMT) is the ideal tool. The main idea of RMT is in fact to isolate generic features. In the past RMT was successfully used to analyse the eigenvalues of the QCD Dirac operator as obtained in lattice gauge calculations. For a recent review see [11]. The perfect agreement between RMT predictions and lattice QCD results led to the general conviction that RMT does apply to QCD, though a rigorous formal proof is still missing.

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<sup>a</sup> e-mail: pepin@daniel.mpi-hd.mpg.de

$$Z(\mu, d, \alpha) = \int \mathcal{D}\mathcal{H} \mathcal{D}\psi_1^\dagger \mathcal{D}\psi_1 \mathcal{D}\psi_2^T \mathcal{D}\psi_2^* \exp \left[ i \begin{pmatrix} \psi_1^\dagger \\ \psi_2^T \end{pmatrix}^T \begin{pmatrix} \mathcal{H} + (D + i\mu)\gamma_0 + im & \eta P_\Delta \\ -\eta^* P_\Delta & -\mathcal{H}^T + (D - i\mu)\gamma_0^T - im \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2^* \end{pmatrix} \right] \quad (2)$$

RMT is especially suited to investigate the chiral phase transition. The Banks-Casher relation

$$\langle q\bar{q} \rangle = -\frac{\pi}{V_4} \rho(\lambda)|_{\lambda=0} \quad (1)$$

relates the chiral condensate  $\langle q\bar{q} \rangle$  to the density  $\rho$  of zero eigenvalues of the Dirac equation and the latter was shown to display universal properties described by RMT. Universal spectral correlations of the QCD Dirac operator can also be computed at finite temperature. In addition to giving exact results for correlations of eigenvalues, RMT can also be used as a schematic model to investigate non-universal quantities.

Recently, Vanderheyden and Jackson have investigated the phase diagram of a QCD-like theory with generic 4-quark couplings in the framework of a RMT model [12, 13] with two quark flavours. Using a saddle point approximation they derived the thermodynamic potential for the quark and diquark condensates and studied the competition between chiral restoration and diquark condensation as a function of temperature and density. They analysed the phase diagram for different values of the coupling constants in the diquark ( $\langle qq \rangle$ ) and chiral ( $\langle q\bar{q} \rangle$ ) channels and showed that the phase diagram can realize a total of six general structures.

To model the effects of temperature, Vanderheyden and Jackson include the lowest Matsubara frequency only. In a more recent work [14], they have also investigated the phase diagram of  $N_c = 2$  QCD with all Matsubara frequencies included. While the inclusion of all Matsubara frequencies does not affect the topology of the phase diagram, it eliminates some unphysical properties like negative baryonic densities at small  $\mu$  and the variation of the chiral condensate with  $\mu$ . The combination of RMT and the Matsubara formalism is, however, only well justified if there are no relevant effects beyond the boundary condition in Euclidian time. QCD just above the phase transition shows, however, strong correlations between the quarks and gluons of a type which can hardly be described by such a simple model. In this contribution, we therefore try to improve the pioneering paper by Vanderheyden and Jackson in this respect by allowing for a more general type of non-random matrix elements. We assume that the properties of the lowest Dirac eigenstates are primarily determined by instanton and anti-instanton field configurations. In the instanton picture [15], spontaneous chiral symmetry breaking is assumed to be generated by randomly distributed uncorrelated instantons/anti-instantons, which allows for a delocalization of the associated quark quasi-zero modes. The restoration of chiral symmetry at high temperature (or high density) can then be realized if the instanton liquid changes from a random ensemble of instantons and anti-instantons to a correlated system where instanton and

anti-instanton pair to form so-called “molecules” [16], the precise definition of which is ambiguous but also irrelevant for our purposes. The formation of such clusters has been observed on the lattice [17] and in numerical simulations of the instanton liquid [18] (for more references on this subject, see also [9]). Of course, other scenarii for chiral symmetry restoration have been proposed; this phase transition can, for example, also be seen as a Mott-Anderson-like transition to an “insulator state” [19]. In this work, however, we assume that the dominant mechanism for chiral symmetry restoration involves instanton molecular correlations. In [20] such a molecular model was used to study the chiral phase transition in the framework of RMT at zero density. The instanton configuration (and therefore the temperature) was characterized by two parameters. It was found that to reduce the value of the chiral condensate by more than a factor two, about 95 percent of the instantons and anti-instantons have to pair in molecules, which confirmed the decisive role played by molecules formation in the restoration of chiral symmetry at finite temperature. In this paper, we want to generalize the results of [20] to finite chemical potential. That will allow us to investigate the stability of the results found in [12, 13] when the temperature effects are not modelled in the most elementary manner by the first Matsubara frequency. Within RMT the properties of the chiral phase transition are determined by the interplay between fluctuations (described by the random matrix elements) and constant terms (for fixed  $T$  and  $\mu$ ) in the Dirac operator. In [12, 13] the latter were assumed to be the same for all states, while we allow for the possibility that they only contribute for a certain fraction of them. We find that for small  $\mu$  the existence of a phase transition is extremely sensitive to this fraction (and thus to the detailed instanton–anti-instanton dynamics), while it becomes nearly independent of it for large  $\mu$ , suggesting that the transition to a diquark condensate is indeed a model-independent feature.

## 2 Formulation of the model

Apart for the parametrization of temperature we follow closely the approach adopted by Vanderheyden and Jackson [12, 13] and refer to their original papers for details of the derivation of the thermodynamic potential. Our starting partition function (for two quark flavours  $\psi_1$  and  $\psi_2$ ) is the following:

*see equation (2) above*

*i.e.*, it is the same as in eq. (1) of [13] but with the substitution  $\pi T \mathbf{1}_N \rightarrow D$  with  $D$  being a  $N \times N$  diagonal matrix, for which a fraction  $\alpha$  of its diagonal elements are equal to a fixed value  $d$  and the rest is zero. This form is motivated

$$\begin{aligned} \Omega(\sigma, \Delta) = & A\Delta^2 + B\sigma^2 - \frac{1}{2} \left\{ \alpha(N_c - 2) \ln \left[ ((\sigma + m + \mu)^2 + d^2)((\sigma + m - \mu)^2 + d^2) \right] + (1 - \alpha)(N_c - 2) \right. \\ & \times \ln \left[ (\sigma + m + \mu)^2(\sigma + m - \mu)^2 \right] + 2\alpha \ln \left[ ((\sigma + m + \mu)^2 + d^2 + \Delta^2)((\sigma + m - \mu)^2 + d^2 + \Delta^2) \right] + 2(1 - \alpha) \\ & \left. \times \ln \left[ ((\sigma + m + \mu)^2 + \Delta^2) \times ((\sigma + m - \mu)^2 + \Delta^2) \right] \right\}. \end{aligned} \quad (6)$$

by the observation that the correlation of instantons and anti-instantons (*i.e.* the formation of molecules) generates such diagonal terms. The size of these terms depends on the anti-instanton–instanton separation and  $d$  has to be interpreted as an average value. If  $\alpha = 1$  (that is, if all instantons and anti-instantons form molecules), one recovers the model of Vanderheyden and Jackson. Let us stress that there are good arguments to assume that for the physical phase transition  $\alpha$  is in fact close to 1. We shall discuss some of them in the conclusion.

In (2),  $m$  is the current mass of the quark and  $\eta$  the source term for the diquark condensate  $\langle \psi_2^T P_\Delta \psi_1 \rangle$ , where  $P_\Delta \equiv iC\gamma_5\lambda_2$  projects on a colour  $\mathbb{3}$ , scalar diquark state ( $C$  is the charge conjugation matrix).  $\eta$  has to be taken to zero at the end of the calculations. A general Hermitian interaction  $\mathcal{H}$  can be written as an expansion over the sixteen Dirac matrices  $\Gamma_K$  times the  $N_c^2$  colour matrices  $\lambda^a$ :

$$\mathcal{H}_{\lambda i \alpha k; \kappa j \beta l} = \sum_{K=1}^{16} (\Gamma_K)_{\lambda i; \kappa j} \sum_{a=1}^{N_c^2} \lambda_{\alpha\beta}^a (A_{\lambda\kappa}^{K a})_{kl}. \quad (3)$$

The measure  $\mathcal{DH}$  associated with the random matrices  $A^{K a}$  is

$$\begin{aligned} \mathcal{DH} = & \left\{ \prod_{K a} \prod_{\lambda\kappa} \mathcal{D}A_{\lambda\kappa}^{K a} \right\} \\ & \times \exp \left[ -N \sum_{K a} \sum_{\lambda\kappa} \beta_K \Sigma_{K a}^2 \text{Tr} [A_{\lambda\kappa}^{K a} (A_{\lambda\kappa}^{K a})^T] \right], \end{aligned} \quad (4)$$

with  $\mathcal{D}A_{\lambda\kappa}^{K a}$  being the Haar measure.  $\beta_K = 1$  if  $K$  corresponds to vector or axial-vector interaction and  $\beta_K = 1/2$  if  $K$  is scalar, pseudo-scalar or tensor. The variance  $\Sigma_{K a}$  is the same for all channels.

Following [12,13], one first performs the integration over the random matrix interaction and then uses a Hubbard-Stratonovitch transformation to introduce two auxiliary variables  $\sigma$  and  $\Delta$ , associated, respectively, with the chiral and diquark condensate. After integrating out the fermion fields, one obtains the following partition function:

$$Z(\mu, \alpha, d) = \int d\sigma d\Delta \exp[-4N\Omega(\sigma, \Delta)] \quad (5)$$

with the thermodynamic potential  $\Omega(\sigma, \Delta)$ :

*see equation (6) above*

As discussed in [12,13], the couplings  $B$  and  $A$  are weighted averages of the Fierz coefficients obtained by projection of the original interaction on chiral and scalar diquark channels, respectively. The ratio  $B/A$  is the only independent parameter and measures the balance between chiral and diquark condensation; varying this ratio allows to explore all the different possible structures of the phase diagram. Imposing the interaction to be Hermitian gives the upper bound  $B/A \leq N_c/2$ . The absolute magnitudes of  $A$  and  $B$  play a secondary role. They introduce a scale for the condensating fields but do not affect the structure of the phase diagram. In the following calculations, we fix  $A = 1$  and vary  $B$ .

To study the various phases for a given value of  $B/A$ , one must minimize the potential  $\Omega$  with respect to both condensates and solve the resulting gap equations. One can already notice that, if  $\alpha \neq 1$ , the second term of this potential is singular at zero density if the chiral condensate vanishes,  $\sigma = 0$  (the current quark mass  $m$  will be set to zero in the following calculations). This is due to the fact that chiral symmetry cannot be completely restored at zero density unless all the instantons and anti-instantons form molecules (*i.e.*  $\alpha = 1$ ), see also [20].

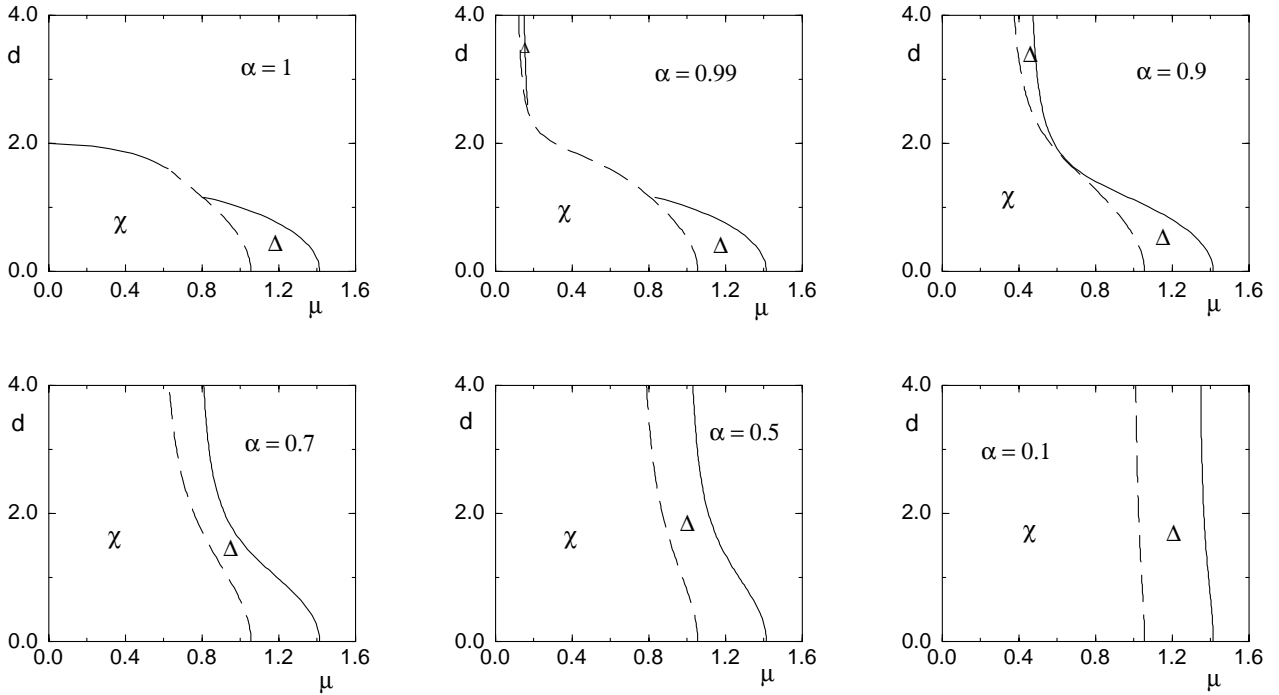
### 3 Results and discussion

The gap equations derived from the potential (6) admit four kinds of solutions: the trivial vacuum, where both condensates  $\sigma$  and  $\Delta$  are zero; the chirally broken phase, where  $\sigma \neq 0$  and  $\Delta = 0$ ; the colour superconducting phase ( $\Delta \neq 0, \sigma = 0$ ); the mixed phase, where both condensates are non-zero. By varying the ratio  $B/A$ , we recovered the various scenarios discussed in [13]; in particular, the mixed phase appears only for  $B/A \geq 1.05$ . We first present results for the coupling ratio  $B/A = \frac{1}{2} \left( \frac{N_c}{N_c - 1} \right) = 0.75$

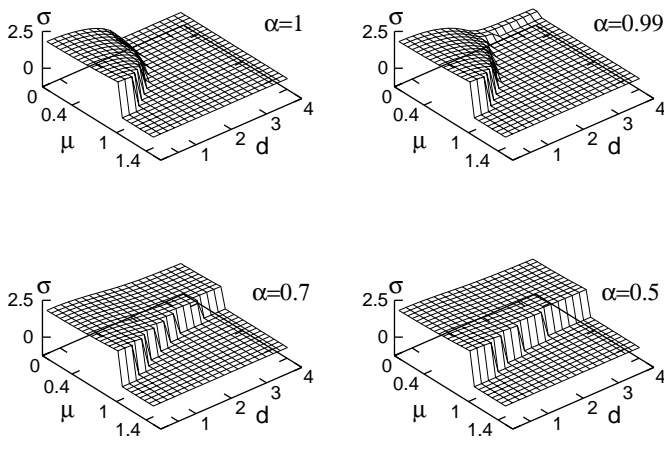
corresponding to one-gluon-exchange as well as to an instanton-induced interaction. Figure 1 shows the phase diagram in the  $(\mu, d)$ -plane for six different values of the molecule fraction  $\alpha$ : 1, 0.99, 0.9, 0.7, 0.5 and 0.1.

Such phase diagrams can be misleading if the actual condensate values are too small to be stable against fluctuations. This does, however, not seem to be the case, as can be seen from figs. 2 and 3.

For  $\alpha = 1$ , one recovers the results of [13] by identifying the parameter  $d$  with the first Matsubara frequency  $\pi T$ . We have checked that the restoration of chiral symmetry is first order for small enough  $d$  ( $d < 1.57$ ). The phase

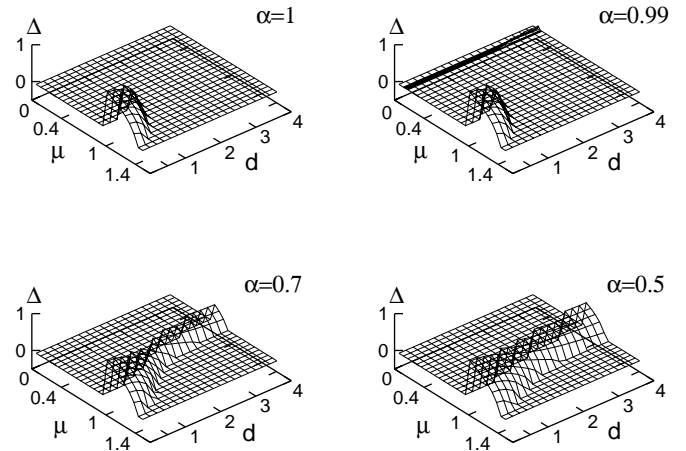


**Fig. 1.** Phase diagram for  $B/A = 0.75$ . The  $x$ -axis corresponds to the chemical potential; the  $y$ -axis to the parameter  $d$ .  $\chi$  and  $\Delta$  label the chiral and diquark phase, respectively. Dashed and continuous lines correspond, respectively, to first- and second-order phase transitions. Note that QCD phenomenology suggests that at the phase transition  $\alpha$  is close to 1.



**Fig. 2.** The strength of the quark condensates  $\sigma$  in fig. 1 ( $B/A = 0.75$ ).

diagram changes noticeably for  $\alpha \neq 1$ : as anticipated previously, chiral symmetry cannot be restored at zero density. (This is well known, as  $\alpha \neq 1$  implies the presence of isolated instantons and thus zero modes.) The chiral transition becomes first order for all values of  $d$ , so there is no longer a tricritical point in the phase diagram. Moreover, the diquark phase appears also at large  $d$  and relatively low density. If  $\alpha$  is further decreased, the diquark phase extends to all values of  $d$  and for growing values of  $d$  the chiral symmetry restoration occurs at higher  $\mu$ . One notes, however, that chiral symmetry restoration always occurs for  $\mu$  above a certain critical value, even if the fraction of



**Fig. 3.** The strength of the diquark condensates  $\Delta$  in fig. 1.

instanton molecules is small. We conclude that for small  $\mu$  the properties of the chiral phase transition depend crucially on the detailed instanton–anti-instanton dynamics. In contrast, the occurrence of a diquark phase seems to be a generic property of the phase diagram whatever the fraction of molecules.

For larger value of the  $B/A$  ratio a mixed phase appears: we show in fig. 4 the results for the case  $B/A = 1.4$ .

Again we reproduce the results of [13] in the case  $\alpha = 1$ . The transition from the mixed phase  $\chi\Delta$  to the diquark phase  $\Delta$  is first order for  $d < 1.1$ . For  $\alpha < 1$ , the changes in the phase diagram are similar to the ones observed in fig. 1. The mixed phase and the diquark phase

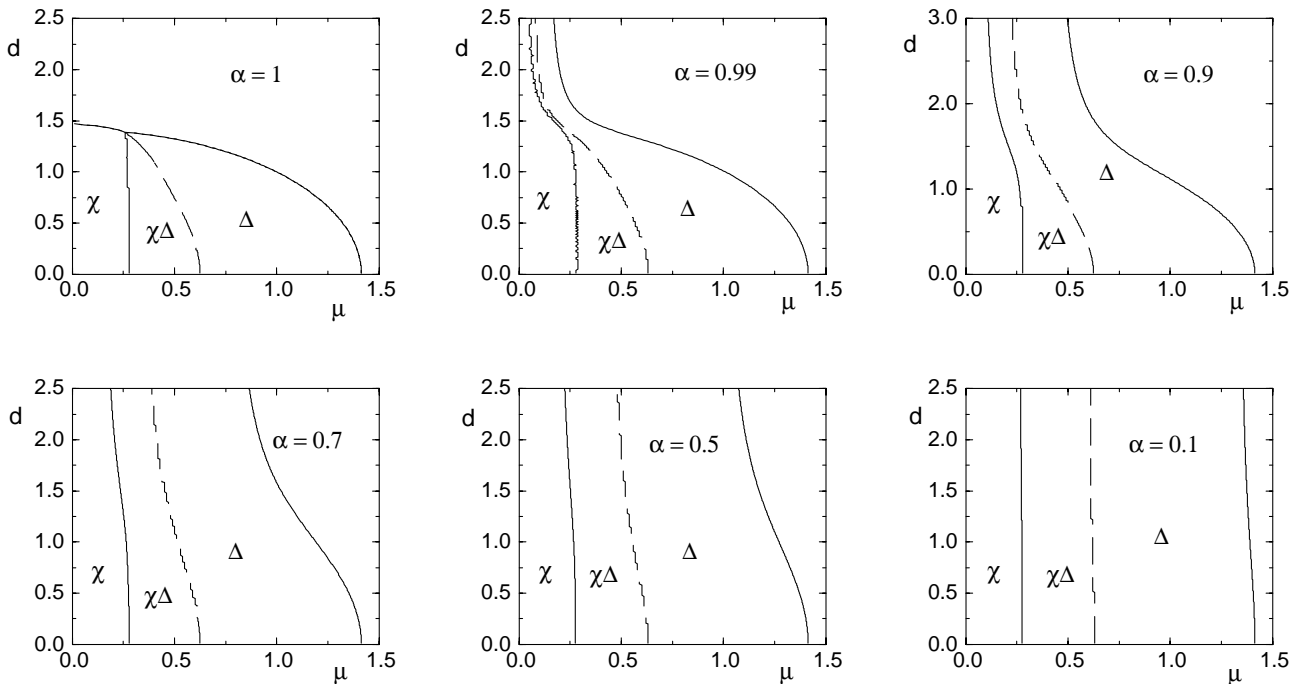


Fig. 4. Same as fig. 1 but for  $B/A = 1.4$ .  $\chi\Delta$  labels the mixed phase.

extend over the whole range of  $d$  and the transition from the mixed phase to the diquark phase is now first order for all values of  $d$ . As in the previous case the diagram is stable only for high enough  $\mu$ .

For completeness, we have also checked the results obtained in [13] when the current quark mass  $m$  is non-zero. For  $\alpha = 1$ , one recovers their phase diagram, where a first-order transition line (corresponding to the first-order chiral transition of the chiral limit) ends in a critical point. This critical point, however, disappears if  $\alpha \neq 1$  and one finds a first-order transition for all  $d$ .

Before we conclude this section, we consider briefly the case  $N_c = 2$ . As has been shown in [12,13], the ratio  $B/A$  is necessarily equal to one in that case. At zero chemical potential, the potential  $\Omega$  depends on the condensation fields  $\sigma$  and  $\Delta$  through the combination  $\sigma^2 + \Delta^2$ . As soon as  $\mu \neq 0$ , this symmetry is broken and the chiral condensate vanishes; the system prefers diquark condensation over chiral symmetry breaking. The same conclusions are valid for a molecule fraction  $\alpha \neq 1$ : the only difference is that the diquark condensate stays non-zero at low  $\mu$  even for large values of  $d$ .

## 4 Conclusions

We have investigated the phase diagram of QCD as a function of density and temperature within Random Matrix Theory along the lines of [12,13]. The aim was to distinguish generic and specific properties. We did not treat the temperature dependence explicitly but encoded it in the fraction  $\alpha(\mu, T)$  and strength  $d(\mu, T)$  of instanton–anti-instanton correlations. We studied the stability of the

phase diagram with respect to variations of  $\alpha$  and  $d$  in comparison with the results of [13]. We found that the low-density part of the phase diagram is highly dependent on  $\alpha(\mu, T)$  and  $d(\mu, T)$  and, thus, on the detailed instanton dynamics, while, on the other hand, the phase diagram is rather stable for high densities. We also found that for finite  $\mu$  not all instantons and anti-instantons have to combine into molecules to allow for a chiral phase transition. These are our main results. Of course, Random Matrix Theory only allows us to derive qualitative results. In particular, it cannot address the question mentioned in the introduction about the dependence of the gap on the QCD coupling constant; this dependence is closely related to the behaviour of the gluon propagators, while our approach amounts to use a contact quark interaction.

For quantitative studies we need realistic models for the functions  $\alpha(\mu, T)$  and  $d(\mu, T)$ , *i.e.* specific input which goes beyond RMT. The fraction of molecules has been calculated as a function of temperature at zero density in [21], where it was shown that the fraction of molecules jumps rapidly from  $\alpha \sim 0.5$  at  $T = 0.7 T_c$  to 1 at  $T_c$ . In refs. [9,22] the molecule density was computed as a function of the chemical potential at zero temperature. It was found that there is a delicate competition between random instantons engaged in diquarks and molecule formation: in fact, the former are dominant and induce the chiral transition. The fraction of atomic instantons goes, however, also in this case, to zero when chiral symmetry is restored. In our case also, the diquark phase appears in the regions of the phase diagram where the effects of molecules are weak enough (meaning either a small fraction  $\alpha$  or a relatively low value of  $d$ ).

Relating our phase diagrams to the one in the  $(\mu, T)$ -plane is not a trivial question. We have shown that the phase diagram is rather unstable at low density, which means that its characteristics in the  $(\mu, T)$ -plane should depend on the details of the functions  $\alpha(\mu, T)$  and  $d(\mu, T)$ . The results quoted above indicate that the fraction of molecules  $\alpha$  changes abruptly at the phase transition. It is natural to assume that this will also happen for the intermediate case ( $T$  and  $\mu$  non-zero), which suggests a large value of  $\alpha$ . Whether this value will be exactly equal to one (which, as we have seen, is necessary for a vanishing of the chiral condensate) or not is not known. However, if the value of  $\alpha$  is large enough ( $> 0.9$ ), the chiral condensate will anyway not be stable against fluctuations. We feel that our present understanding of the QCD phase transition is insufficient for reliable quantitative models for the functional form of  $\alpha(\mu, T)$  and  $d(\mu, T)$ .

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